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LETTER TO THE EDITOR

Relation between the Gelfand–Levitan procedure and the method of supersymmetric partners

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Abstract. It is shown that, for one-dimensional quantum mechanical systems, the Gelfand-Levitan procedure based on an inverse scattering approach is equivalent to two successive applications of the method based on construction of supersymmetric partners.

In the literature two methods have been discussed for constructing from a Hamiltonian H for a one-dimensional quantum mechanical system another Hamiltonian H' such that the spectrum of H' consists of all eigenvalues of H except either (i) the ground state of H is removed or (ii) a state below the ground state of H is added. The first method is the Gelfand-Levitan (GL) procedure based on the inverse scattering method (Abraham and Moses 1980). The second method makes use of the idea of constructing supersymmetric partners (Sukumar 1985a), which we shall henceforth refer to as the SUSY method. (We note that the SUSY method is essentially the same as that due to Darboux (Darboux 1882, Bateman 1966) for constructing hierarchies of second-order differential equations with known exact solutions.) The two methods yield different forms for H'. Our aim here is to relate the two. In particular, we show that the H' given by the GL method can be obtained by two successive applications of the SUSY method. A similar investigation has been carried out by Sukumar (1985b) in the context of the radial Schrödinger equation.

Consider a one-dimensional quantum mechanical system with the Hamiltonian

$$H=-(\mathrm{d}^2/\mathrm{d}x^2)+V(x) \qquad -\infty<x<\infty. \tag{1}$$

Let E_0 and ψ_0 denote the ground-state energy and the corresponding normalised eigenfunction be as follows:

$$H\psi_0=E_0\psi_0 \qquad \int_{-\infty}^{+\infty} \mathrm{d}x\,\psi_0^2(x)=1. \tag{2}$$

For simplicity let us put $E_0=0$. From H in (1) we want to construct a Hamiltonian H' whose spectrum is the same as that of H except that the state at $E=0$ is absent. Application of the GL procedure yields the following expression for H':

$$H'=-(\mathrm{d}^2/\mathrm{d}x^2)+V'(x) \tag{3}$$

where

$$V'(x)=V(x)-2(\mathrm{d}^2/\mathrm{d}x^2)\ln\left(1-\int_{-\infty}^{x}\mathrm{d}y\,\psi_0^2(y)\right). \tag{4}$$

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Let us now consider the situation when $E_0 \neq 0$ and we want to construct a H' with the same spectrum as H plus an added state at $E_0 = 0$. Applying the GL procedure the desired Hamiltonian H' is found to be

$$H' = -(\mathrm{d}^2/\mathrm{d}x^2) + V'(x) \tag{5}$$

where

$$V'(x) = V(x) - 2(\mathrm{d}^2/\mathrm{d}x^2)\ln\left(1 + \int_{-\infty}^{x} \mathrm{d}y\, \psi_0^2(y)\right) \tag{6}$$

with ψ_0 in (6) denoting the formal eigenfunction of H at $E = 0$.

We now turn our attention to the SUSY method. First we consider deletion of the ground state $E_0 = 0$. From (2) we can express $V(x)$ in terms of ψ_0 as

$$V(x) = \frac{1}{\psi_0}\frac{\mathrm{d}^2\psi_0}{\mathrm{d}x^2}. \tag{7}$$

Substituting for $V(x)$ from (7) in (1) we can factorise H as follows:

$$H = \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0\right) \equiv Q^\dagger Q. \tag{8}$$

Note that this factorisation is not unique. We can replace ψ_0 in (8) by

$$\psi_0 \to \psi_0\left(1 + \alpha\int_{-\infty}^{x} \mathrm{d}y \frac{1}{\psi_0^2(y)}\right) \tag{9}$$

which is the general solution of the equation $H\psi = 0$. This fact will be used later. The supersymmetric partner H' of H is obtained by

$$\begin{aligned} H' = QQ^\dagger &= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_0\right) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{1}{\psi_0}\right)\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{1}{\psi_0}\right)\right). \end{aligned} \tag{10}$$

The spectrum of H' is identical to that of H with the exception of the state at $E = 0$. The potential $V'(x)$ in H' is related to $V(x)$ through the following relation:

$$V'(x) = V(x) - 2(\mathrm{d}^2/\mathrm{d}x^2)\ln\psi_0 \tag{11}$$

which is different from the corresponding relation (4) of the GL method.

Note that H' in (10) is obtained from H given in (8) simply by making the transformation $\psi_0 \to 1/\psi_0$. The process of deletion of the ground state in the SUSY method can be understood as follows. The transformation $\psi_0 \to 1/\psi_0$ takes us from H to H' which has a formal eigenfunction $1/\psi_0$ at $E = 0$. Since ψ_0 is normalisable and $1/\psi_0$ is not, the effect of the transformation is simply to delete this state from the spectrum of H'. All other states remain unaffected. The same logic can be extended to the process of adding to or maintaining the spectrum of H. Consider, for instance, the case when H has a ground state at $E_0 \neq 0$ and we want to add a state at $E_0 = 0$. Let ψ_0 denote the formal eigenfunction of H corresponding to $E = 0$. Then factorising H in terms of ψ_0 and making the transformation $\psi_0 \to 1/\psi_0$, we can construct a H' which has an eigenfunction at $E = 0$. If $1/\psi_0$ is normalisable, then H' would have a spectrum identical to that of H with an additional state at $E = 0$. If, however, $1/\psi$ is not normalisable then the spectrum remains unchanged. These ideas, together with the freedom to replace a solution ψ_0 by the more general solution $\psi_0(1+\alpha\int_{-\infty}^{x} dy\,(1/\psi_0^2(y)))$ in the factored expression for H in terms of ψ_0, are the two basic ingredients which allow us to establish a relation between the two methods for addition and deletion of a state as we shall see.

First consider the case of deletion of a state. Applying $\psi_0 \to 1/\psi_0$ to H in (8) we obtain H', as in (10). We now replace $1/\psi_0$ in (10) by

$$\chi = \frac{1}{\psi_0}\left(1+\alpha\int_{-\infty}^{x} dy\,\psi_0^2(y)\right). \tag{12}$$

This does not alter H' in any way, so that

$$H' = \left(\frac{d}{dx}+\frac{d}{dx}\ln\chi\right)\left(-\frac{d}{dx}+\frac{d}{dx}\ln\chi\right). \tag{13}$$

Now we construct, by $\chi \to 1/\chi$, the supersymmetric partner H'' of H' as

$$H'' = \left(\frac{d}{dx}+\frac{d}{dx}\ln\left(\frac{1}{\chi}\right)\right)\left(-\frac{d}{dx}+\frac{d}{x}\ln\left(\frac{1}{\chi}\right)\right) \tag{14}$$

and $1/\chi$ satisfies

$$H''(1/\chi) = 0. \tag{15}$$

The spectrum of H'' depends on the normalisation of $1/\chi$, i.e.

$$\int_{-\infty}^{\infty} dx\,\frac{1}{\chi^2(x)} = \int_{-\infty}^{\infty} dx\,\psi_0^2(x)\left(1+\alpha\int_{-\infty}^{x} dy\,\psi_0^2(y)\right)^{-2} = \frac{C}{1+\alpha C} \tag{16}$$

where C denotes the normalisation of ψ_0. If ψ_0 is assumed to be normalised to unity we find that the choice $\alpha=-1$ yields a non-normalisable $1/\chi$. Hence it is clear from the earlier discussion that with $\alpha=-1$ the effect of $\chi\to 1/\chi$ will be to maintain the spectrum of H'. That is, H'' has the same spectrum as H with the state $E=0$ removed. It is easy to check that the Hamiltonian H'' has the form

$$H'' = -\frac{d^2}{dx^2}+V(x)-2\frac{d^2}{dx^2}\ln\left(1-\int_{-\infty}^{x} dy\,\psi_0^2(y)\right) \tag{17}$$

which is the same as the Hamiltonian obtained by the GL method. Thus the GL method for deletion of the ground state can be analysed in terms of two successive applications of the SUSY method.

Consider the case of addition of a state at $E=0$. The analysis is exactly the same as is the case of deletion, with the proviso that ψ_0 denotes the formal eigenfunction of H at $E=0$, i.e. ψ_0 is not normalisable. After obtaining H' by replacing ψ_0 by $1/\psi_0$ we rewrite H' in terms of χ as in (13) and make the transformation $\chi\to 1/\chi$. This time, since ψ_0 is not normalisable, C in (16) is infinite so that $1/\chi$ is normalised to $1/\alpha$. Choosing $\alpha=+1$ we find that H'' has an eigenstate at $E=0$ normalised to unity. Hence H'' has a spectrum identical to that of H with a state added at $E=0$. It is easily seen that H'', obtained with $\alpha=+1$, is identical to (6) obtained by the GL method for addition of a state. Note that the relation between the spectra of H and H'' is independent of the normalisability of $1/\psi_0$. If $1/\psi_0$ is normalisable, then H' already has a state at $E=0$ and $\chi\to 1/\chi$ simply maintains that spectrum of H'. If $1/\psi_0$ is not normalisable then H' also does not have a state at $E=0$, but $\chi\to 1/\chi$ adds a state at $E=0$. In either case, the resulting H'' has a state at $E=0$ and is identical to that obtained by the GL method. Thus the GL method for addition of a state at $E=0$ can also be analysed in terms of two successive applications of the SUSY method.

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